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Rings with the Double Centralizer Property*

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Let M be a left unital module over a ring R; write $\mathscr{C} = \operatorname{end}({}_RM)$ and, considering M as a $\mathscr{C}$-module, $\mathscr{D} = \operatorname{end}(M_{\mathscr{C}})$. The module M is said to have the double centralizer property, if the canonic homomorphism of R to $\mathscr{D}$ is onto. If every finitely generated faithful left R-module has the double centralizer property, the ring R is called a left $QF - 1$ ring. This concept was introduced by Thrall [13] as a generalization of quasi-Frobenius rings. To our knowledge, no characterization of left $QF - 1$ rings in terms of their ring structure is known. A ring R is said to be left-balanced, if every left R-module has the double centralizer property. It is well known that every artinian uniserial ring is both left and right balanced (Nakayama [9]; Nesbitt and Thrall [11]). Recently, several authors proved the converse for commutative rings (Dickson and Fuller [2]; Camillo [1]) and Jans in [7] for finite-dimensional algebras over algebraically closed fields. In the same paper, Jans conjectured that the converse was true in general.

The aim of the present paper is to describe the structure of balanced rings and of certain local $QF - 1$ rings. The first result in this direction is the following theorem proved previously by Camillo [1] for commutative rings.

THEOREM A. *A left balanced ring is left artinian.*

Since every left artinian left balanced ring is a finite direct sum of full matrix rings over left artinian left balanced local rings (Fuller [6]), the structure theorem on balanced rings can be formulated for local rings.

THEOREM B. *Let R be a left artinian local ring with the radical W such that R/W^3 is left-balanced. Then either*

(i) *R is left uniserial; or*

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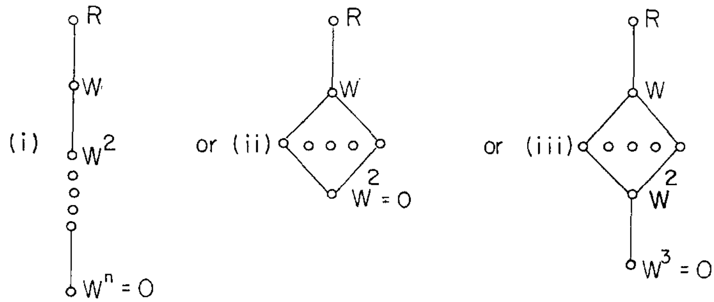
(ii) $W^2 = 0$, $\dim({}_{R/W}W) = 2$ *and, for any two nonzero elements* x, y *of* W, $Rx + yR = W$; *or*

(iii) $W^3 = 0$, W^2 *is the unique minimal left ideal and* R/W^2 *is a ring described in* (ii).

Hence, the left ideal structure of a local left-balanced ring can be illustrated as follows:



Making use of Theorem B, one can deduce

THEOREM C. *Let R be a ring finitely generated over its center. Then R is left-balanced if and only if it is uniserial* (*in the sense of Nakayama*).

Thus, in particular, such a ring is left balanced if and only if it is right balanced. Theorem C proves Jans' conjecture for a large class of rings including finite-dimensional algebras over arbitrary fields. However, the conjecture is not true in general. It is shown in [5] that the assumption on R to be finitely generated over its center is necessary: Indeed, there are examples of balanced rings which satisfy the condition (ii) of Theorem B. On the other hand, the question on whether balanced rings of the type (iii) of Theorem B exist is open.

The proofs of the theorems above depend on investigations of artinian local $QF-1$ rings. In particular, the following result is proved.

THEOREM D. *Let R be an artinian local ring finitely generated over its center. Then R is a* $QF-1$ *ring if and only if R is quasi-Frobenius.*

For commutative rings, this was proved by Dickson and Fuller [2]. Again, the examples of [5] show that the condition on R to be finitely generated over its center is necessary.

After introducing notation and terminology in the first section of the paper, Section 2 contains several constructions of modules which are not balanced. These are afterwards employed in the proof of Theorem A in Section 3. The following Section 4 consists of some further constructions of nonbalanced modules to be used in Section 5 to prove Theorem B. The final Section 6 constitutes the proof of Theorem C and Theorem D.

1. Preliminaries

Throughout the paper, R denotes an (associative) ring with unity. By an R-module we always understand a unital R-module; the symbols ${}_RM$ or M_R will be used to underline the fact that M is a left or right R-module, respectively. Given an R-module M, denote by $\operatorname{rad} M$ the intersection of all maximal submodules of M if there are any; otherwise, $\operatorname{rad} M = M$. Dually, if M has minimal submodules, $\operatorname{soc} M$ denotes their union; if M has no minimal submodules, $\operatorname{soc} M = 0$. Thus, considering a ring R as a left R-module (or a right R-module) we get the concept of the *left socle* (or the *right socle*) of R, as well as the concept of the *radical* of R; the latter will be denoted consistently by W. It is immediate to see that $WM \subseteq \operatorname{rad} M$ for any left R-module M. If $\operatorname{rad} M$ is the only proper maximal submodule of M, M will be called *local*. Thus, all local modules are *monogenic*. And, if ${}_RR$ (and, for the matter R_R) is local, then R is said to be a local ring. If M has a composition series, denote by $\partial(M)$ its length; again, in case of an artinian ring R, we can speak of *left length* $\partial_l(R)$ and *right length* $\partial_r(R)$ of R.

Let M be a left R-module M, $\mathscr{C}(M) = \operatorname{end}({}_RM)$ the *centralizer* and $\mathscr{D}(M) = \operatorname{end}(M_{\mathscr{C}})$ the *double centralizer* of M. Throughout the paper, the elements φ of $\mathscr{C}(M)$ will act on M from the right, the elements Ψ of $\mathscr{D}(M)$ will act from the left; in particular, $\Psi(m\varphi) = (\Psi m)\varphi$ for all $m \in M$. The multiplication of these elements will also be written in the respective order.

Following Bass, a ring R is said to be *right perfect* if W is T-nilpotent and R/W artinian. And R is right perfect if and only if R/W is artinian and every quotient ring of R has a nonzero left socle (see, e.g., [4]); in such a ring, $W^i \neq 0$ $(i \geqslant 1)$ necessarily implies that $W^i \neq W^{i+1}$ (cf. [3]). The concept of T-nilpotence has been weakened by Camillo [1] to that of bi-T-nilpotence: W is said to be *bi-T-nilpotent* if, for every sequence $\{w_i\}$, $w_i \in W$, indexed by all integers i, there are $i_1 \geqslant 0 \geqslant i_2$ such that $w_{i_1} w_{i_1-1} \cdots w_{i_2+1} w_{i_2} = 0$. Let us remark that by a *perfect* ring we shall understand a ring which is both right and left perfect.

A module is said to be *uniserial* if all its submodules form a chain with respect to inclusion. Hence, a *left* (or *right*) *uniserial* ring is necessarily local. Following Nakayama [9], a *uniserial ring* is defined to be a finite direct sum of full matrix rings over artinian local rings which are both left and right uniserial. It is not difficult to see that R is uniserial if and only if R/W^2 is uniserial (cf. [9]).

For an element m of a left R-module M, the *annihilator* (order) $\{r \mid r \in R$ such that $rm = 0\} \subseteq R$ of m will be denoted by $\text{ann}(m)$; also, given a submodule N of M, write $\text{ann}(N) = \bigcap_{m \in N} \text{ann}(m)$. Correspondingly, in a ring R, we have the concepts of left and right annihilators (denoted by ann_l and ann_r, respectively). An artinian ring in which $\text{ann}_l(\text{ann}_r(L)) = L$ and $\text{ann}_r(\text{ann}_l(K)) = K$ for every left ideal L and every right ideal K is called *quasi-Frobenius*. As an immediate consequence, $\partial_l(R) = \partial_r(R)$. The concept of a quasi-Frobenius ring has been introduced by Nakayama in [9]; he has also shown that an artinian local ring is quasi-Frobenius if and only if $\partial_l(\text{soc }_R R) = \partial_r(\text{soc } R_R) = 1$. Moreover, an artinian ring R is uniserial if and only if every quotient ring of R is quasi-Frobenius (see [10]).

Later, Thrall [13] has generalized the concept of a quasi-Frobenius ring to that of a $QF-1$ ring: A ring R is said to be a *left* (or *right*) $QF-1$ *ring* if every finitely generated faithful left (or right) R-module is balanced. Here, an R-module M is called *balanced* (or to have the double centralizer property) if all elements of its double centralizer are induced by the ring multiplication. If every left (or right) R-module is balanced, R is said to be *left* (or *right*) *balanced.* And, again, by a balanced ring, or $QF-1$ ring, we shall mean a ring which is both left and right balanced, or left and right $QF-1$, respectively.

2. Constructions of Nonbalanced Modules

In this section, we have collected several constructions of modules which are not balanced; these constructions are essential for all our theorems on balanced rings.

In the first construction, two copies of a local ring R considered as a left R-module are amalgamated over isomorphic left ideals. This method of constructing nonbalanced modules was used previously by several authors (Dickson and Fuller [2]; Jans [7]); their results can be obtained easily from the following more general

Construction I. *Let R be a local ring with a minimal right ideal. Let U_1, U_2 be two nonzero isomorphic left ideals and I_1, I_2 two two-sided ideals of R such that*

$$U_i \subseteq I_i \ (i = 1, 2) \qquad \textit{and} \qquad I_1 \cap I_2 = 0.$$

Then there is a finitely generated faithful left R-module which is not balanced.

Proof. Let

$$M = (R \oplus R)/D,$$

where $D = \{(d, -d\xi) \mid d \in U_1\}$ with an isomorphism $\xi\colon U_1 \to U_2$. Obviously, M is finitely generated and faithful. Every endomorphism of M can be lifted to an endomorphism of ${}_R(R \oplus R)$ and, in this way, we get just those endomorphisms of the left module $R \oplus R$ which map D into D. Let

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

be the matrix representation of such an endomorphism of $R \oplus R$; here, α_{ij} denote endomorphisms of ${}_RR$, that is to say, right multiplications by elements of R. For $(d, -d\xi) \in D$, we get

$$(d, -d\xi)\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = (d\alpha_{11} - (d\xi)\,\alpha_{21}\,, d\alpha_{12} - (d\xi)\,\alpha_{22});$$

in order that this element lies in D, it is necessary that

$$d\alpha_{1i} - (d\xi)\alpha_{2i} \in U_i \qquad (i = 1, 2).$$

This implies that $\alpha_{21} \in W$ and $\alpha_{12} \in W$ with W denoting the radical of R. For, if $\alpha_{21} \notin W$, then $\alpha_{21}^{-1} \in R$ exists and we get

$$U_2 = U_1\xi \subseteq U_1\alpha_{21}^{-1} + U_1\alpha_{11}\alpha_{21}^{-1} \subseteq I_1\,,$$

a contradiction. Similarly, if $\alpha_{12} \notin W$, then

$$U_1 \subseteq U_2\alpha_{12}^{-1} + (U_1\xi)\,\alpha_{22}\alpha_{12}^{-1} = U_2\alpha_{12}^{-1} + U_2\alpha_{22}\alpha_{12}^{-1} \subseteq I_2\,,$$

again a contradiction.

Now, we are going to construct an additive homomorphism $\Psi\colon R \oplus R \to R \oplus R$ commuting with all matrices $\left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right)$, where $\alpha_{21} \in W$ and $\alpha_{12} \in W$, and mapping D into itself. Thus, Ψ will induce an element of the double centralizer $\mathscr{D}(M)$ of M. Take a nonzero element z of a minimal right ideal and define Ψ by

$$\Psi(x, y) = (zx, 0) \qquad \text{for all} \quad (x, y) \in R \oplus R.$$

Evidently, since $U_1 \subseteq W$, $\Psi(D) = 0$. An easy calculation yields

$$[\Psi(x, y)]\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = (zx\alpha_{11}\,, zx\alpha_{12}) = (zx\alpha_{11}\,, 0),$$

because, again, z belongs to the right socle and $x\alpha_{12} \in W$. Similarly,

$$\Psi\left[(x, y)\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}\right] = (zx\alpha_{11} + zy\alpha_{21}\,, 0) = (zx\alpha_{11}\,, 0).$$

Thus, Ψ induces an element of $\mathscr{D}(M)$.

Assume that this morphism is induced by an element $\rho \in R$. Then, necessarily,

$$\Psi(x, y) - (\rho x, \rho y) \in D \qquad \text{for all} \quad (x, y) \in R \oplus R.$$

Hence, if $(x, y) = (0, 1)$, we get $(0, \rho) \in D$ and, consequently, $\rho = 0$. But then, for $(x, y) = (1, 0)$, we have $(z, 0) \in D$, a contradiction. Thus, the morphism defined by Ψ is not induced by the ring multiplication, i.e., M is not balanced.

Construction I implies, that a perfect local $QF - 1$ ring has a unique minimal two-sided ideal. If R is commutative, then R has to be a quasi-Frobenius ring; this is the main result of Dickson and Fuller in [2]. Also, if R is a finite-dimensional algebra over an algebraically closed field, and $W^2 = 0$, then R has to be uniserial (because every left ideal is two-sided). This yields Jans' result of [7] that an arbitrary (not necessarily local) finite-dimensional left balanced algebra over an algebraically closed field is uniserial. Let us point out that these results will not be needed in the sequel; they are mentioned here briefly just to illustrate the extent of applicability of Construction I.

Under certain conditions, the direct sum of two modules M_1 and M_2 can be shown to be nonbalanced; such conditions were given, e.g., by Morita [8] and Camillo [1]. Another sufficient condition is that both modules be local and faithful and none of them be a quotient of the other. This follows from the following more general construction which will be required later.

CONSTRUCTION II. *Let R be a local ring with the radical W. Let M_1 and M_2 be two left R-modules such that, for $i \neq j$, every homomorphism φ: $M_i \to M_j$ satisfies $M_i\varphi \subseteq WM_j$. Let, moreover,* $\operatorname{soc} R_R \not\subseteq \operatorname{ann}(M_1)$ *and M_2 be faithful. Then $M = M_1 \oplus M_2$ is a faithful R-module which is not balanced.*

Proof. Let us represent the elements of the centralizer of M by the matrices

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}, \qquad \text{where} \qquad \varphi_{ij} : M_i \to M_j .$$

Take an element $z \in \operatorname{soc} R_R \backslash \operatorname{ann}(M_1)$ and define an additive homomorphism Ψ: $M \to M$ by

$$\Psi(m_1, m_2) = (zm_1, 0) \qquad \text{for all} \quad (m_1, m_2) \in M_1 \oplus M_2 .$$

In fact, Ψ belongs to the double centralizer of M because

$$\begin{aligned} &[\Psi(m_1, m_2)] \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \\ &\quad = (zm_1, 0) \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} = (zm_1\varphi_{11}, zm_1\varphi_{12}) \\ &\quad = (zm_1\varphi_{11}, 0) = (zm_1\varphi_{11} + zm_2\varphi_{21}, 0) = (z(m_1\varphi_{11} + m_2\varphi_{21}), 0) \\ &\quad = \Psi(m_1\varphi_{11} + m_2\varphi_{21}, m_1\varphi_{12} + m_2\varphi_{22}) = \Psi\left[(m_1, m_2) \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}\right]; \end{aligned}$$

here, $zm_1\varphi_{12} = 0$ and $zm_2\varphi_{21} = 0$ in view of the fact that $m_i\varphi_{ij} \in WM_j$, for $i \neq j$. Finally, Ψ is not induced by left multiplication. For, assuming that $\Psi(m_1, m_2) = (\rho m_1, \rho m_2)$ for a certain $\rho \in R$, we deduce that $\rho = 0$ since M_2 is faithful. But since $z \notin \operatorname{ann}(M_1)$, there is $m_1' \in M_1$ such that $zm_1' \neq 0$ and thus

$$\Psi(m_1', 0) = (zm_1', 0) \neq (\rho m_1', 0).$$

As a consequence, M is not balanced.

Construction II will be used in the proof of the next construction which deals with a situation similar to that of Construction I. Here, we are going to replace the condition that the left ideals U_i are contained in disjoint two-sided ideals by an asymmetric assumption on U_1 and U_2 and a condition on the right socle.

CONSTRUCTION III. *Let R be a local ring. Let U_1, U_2 be two nonzero left ideals and I_1 a two-sided ideal of R such that*

$$U_1 \subseteq I_1 \qquad \text{and} \qquad I_1 \cap U_2 = 0.$$

Let, furthermore, U_2 contain no nonzero two-sided ideal of R and $\operatorname{soc} R_R \not\subseteq U_1$. *Then there is a finitely generated faithful left R-module which is not balanced.*

Proof. Let $M_i = R/U_i$ and $M = M_1 \oplus M_2$. First, obviously, M_2 is faithful and, since $\operatorname{ann}(M_1) \subseteq U_1$, $\operatorname{soc} R_R \not\subseteq \operatorname{ann}(M_1)$. In order to be able to apply Construction II, we have to look at the morphisms between M_1 and M_2.

Every homomorphism φ_1: $M_1 \to M_2$ can be lifted to an endomorphism of ${}_RR$ mapping U_1 into U_2. Thus, there is an element $\alpha_1 \in R$ (operating on ${}_RR$ by right multiplication) with $U_1\alpha_1 \subseteq U_2$, and therefore

$$U_1\alpha_1 \subseteq I_1 \cap U_2 = 0.$$

This means that $\alpha_1 \in W$, and hence $M_1\varphi_1 \subseteq WM_2$.

Similarly, every homomorphism φ_2: $M_2 \to M_1$ can be lifted to the right multiplication by α_2 on ${}_RR$ satisfying $U_2\alpha_2 \subseteq U_1$. Again $\alpha_2 \in W$; for, if α_2 were a unit, then

$$U_2 \subseteq U_1\alpha_2^{-1} \subseteq I_1,$$

contradicting our hypothesis. Consequently, $M_2\varphi_2 \subseteq WM_1$ and Construction II can be applied.

In order to show that the length of the left socle and the right socle of a balanced local ring is bounded, we need yet another construction. Here, we show that some quotient modules of a certain ring R are not balanced.

CONSTRUCTION IV. *Let R be a local ring. Let $U \subseteq \operatorname{soc} R_R$ be a nonzero left ideal containing no nonzero two-sided ideal. Let r be a unit of R such that*

$$Ur \nsubseteq U \qquad \textit{and} \qquad \operatorname{soc} R_R \nsubseteq U + Ur.$$

Then there is a finitely generated faithful left R-module which is not balanced.

Proof. Let $M = R/U$; for every $x \in R$, write

$$\bar{x} = x + U \in M.$$

Clearly, M is a finitely generated faithful left R-module.

Denote by $\mathscr{C}$ the centralizer of M. The elements φ of $\mathscr{C}$ can be lifted to endomorphisms of R and, in this way, we get just those elements $\alpha_\varphi \in R$ (acting on ${}_RR$ by right multiplication) which satisfy $U\alpha_\varphi \subseteq U$. Thus, denoting by W, $T = W/U$ and $\mathscr{W}$ the radicals of R, M and $\mathscr{C}$, respectively, we deduce from here that

$$\mathscr{W} = \{\varphi \mid \varphi \in \mathscr{C} \text{ and } \alpha_\varphi \in W\};$$

hence, $\mathscr{C}$ is local and $M\mathscr{W} \subseteq T$. Evidently, T is a $\mathscr{C}$-submodule of M and the $\mathscr{C}$-module M/T is completely reducible. In fact, we can show that

$$M/T = (\bar{1} + T)\mathscr{C} \oplus (\bar{r} + T)\mathscr{C} \oplus C$$

with a suitable complement C. This follows from the fact that, in view of the assumptions put on r,

$$\bar{1}\mathscr{C} \cap \bar{r}\mathscr{C} \subseteq T.$$

Indeed, assuming the contrary, there would be $\varphi \in \mathscr{C}$ such that $\bar{1}\varphi - \bar{r} \in T$ and, lifting φ to an endomorphism α_φ of ${}_RR$, we would get $1\alpha_\varphi - r \in W$. However, $\alpha_\varphi - r \in W$ together with $U \subseteq \operatorname{soc} R_R$ imply $U(\alpha_\varphi - r) = 0$, and thus $U\alpha_\varphi = Ur \nsubseteq U$, contradicting the fact that α_φ induces the endomorphism φ of M.

Now, according to our assumptions on U, there is an element $z \in \operatorname{soc} R_R \backslash (U + Ur)$. Obviously, since $\bar{z}\mathscr{W} = 0$, $\bar{z}$ belongs also to the socle of the $\mathscr{C}$-module M. We are going to construct an element Ψ of the double centralizer $\mathscr{D}$ of M such that $\Psi(\bar{1}) = \bar{0}$ and $\Psi(\bar{r}) = \bar{z}$. First, define the $\mathscr{C}$-homomorphism Ψ': $(M/T)_\mathscr{C} \to \operatorname{soc}(M_\mathscr{C})$ by

$$\Psi'(\bar{1} + T) = \bar{0}, \quad \Psi'(\bar{r} + T) = \bar{z} \quad \text{and} \quad \Psi'(\bar{x} + T) = \bar{0} \quad \text{for} \quad \bar{x} + T \in C,$$

and then put

$$\Psi = \iota\Psi'\epsilon,$$

where ϵ: $M_{\mathscr{C}} \to (M/T)_{\mathscr{C}}$ is the canonic epimorphism and ι: $\operatorname{soc}(M_{\mathscr{C}}) \to M_{\mathscr{C}}$ is the inclusion. Obviously, $\Psi \in \mathscr{D}$ has the required properties.

In order to complete the proof, it is sufficient to show that Ψ is not induced by left multiplication. Thus, assume that there is an element $\sigma \in R$ such that

$$\sigma\bar{x} = \Psi(\bar{x}) \qquad \text{for all} \quad \bar{x} \in M.$$

Then, $\sigma\bar{1} = \bar{0}$ implies $\sigma \in U$ and $\sigma\bar{r} = \bar{z}$ implies $z \in \sigma r + U$. Hence, combining both implications we get a contradiction of our assumption that $z \notin U + Ur$.

We conclude that M is not balanced.

3. Left-Balanced Rings Are Left Artinian

The main purpose of this section is to prove the statement in its title. In order to facilitate the proof, we are going to derive some preliminary structural properties of left artinian local rings. These results will also be used throughout Section 5.

Lemma 3.1. *Let R be a local right perfect left $QF - 1$ ring with a minimal right ideal. Then R has a unique minimal two-sided ideal I and, moreover, either*

(i) $\partial_1(I) = 1$ *and I is the left socle of R, or*

(ii) $\partial_1(I) = 1$ *and I is the right socle of R, or*

(iii) $\partial_1(I) = 2$ *and I is both the left and the right socle of R.*

Proof. Write $S = \operatorname{soc} R_R$ and assume that there is a two-sided ideal $I \subseteq S$ with $\partial_1(I) = 1$. In this case, we claim that I is the unique minimal two-sided ideal and that I is either the left or the right socle of R. The first statement is an immediate consequence of Construction I and the statement on the socles follows from Construction III. Indeed, if I is neither the left nor the right socle of R, we take a minimal left ideal U_2 which is not contained in I and $U_1 = I$ to satisfy the assumptions of Construction III.

Now, if no two-sided ideal of length 1 in S exists, denote by I the left socle of S; thus $\partial_1(I) \geqslant 2$. Obviously, I is a unique minimal two-sided ideal and it is the left socle of R; for, otherwise, we can apply again Construction I or Construction III to obtain a contradiction. Furthermore, making use of Construction IV for a minimal left ideal $U \subseteq I$, we deduce that $I = S$ and that $\partial_1(I) \leqslant 2$. Lemma 3.1 follows.

As a simple consequence, we can formulate

Corollary 3.2. *Let R be a local left $QF - 1$ ring with the radical W satisfying $W^2 = 0$. Then $\dim({}_{R/W}W) \leqslant 2$ and W is the minimal two-sided ideal of R.*

LEMMA 3.3. *Let R be a left balanced local ring with the radical W. If $W^n = 0$, then the powers W^ν are the only two-sided ideals of R and, for each $W^\nu \neq 0$, $W^\nu/W^{\nu+1}$ is both the left and the right socle of $R/W^{\nu+1}$; moreover, $\partial_l(W^\nu/W^{\nu+1}) \leqslant 2$. Thus, in particular, R is left artinian.*

Proof. First, in view of Construction I, R is obviously two-sided uniserial. Moreover, if, for some ν, a two-sided ideal I exists such that $W^{\nu+1} \subsetneqq I \subsetneqq W^\nu$, then Construction III applied to $R/W^{\nu+1}$ yields a contradiction.

Now, taking a fixed ν with $W^\nu \neq 0$, consider the ring $R/W^{\nu+1}$. Without loss of generality, we can suppose that $W^{\nu+1} = 0$. Observe that both the left socle S_l and the right socle S_r of R contain W^ν which is the minimal two-sided ideal of R. According to Lemma 3.1, either $S_l = W^\nu = S_r$, or $S_l = W^\nu \subsetneqq S_r$, or $S_l \supsetneqq W^\nu = S_r$. Assuming that $W^\nu \subsetneqq S_r$, we deduce that $W^{\nu-1} \subseteqq S_r$; for, the powers of W are the only two-sided ideals of R (here, $W^{\nu-1} = R$ for $\nu = 1$). And hence, we get a contradiction, because

$$W^\nu = W^{\nu-1} \cdot W \subseteqq S_r \cdot W = 0.$$

Similarly, we can verify that the case $S_l \supsetneqq W^\nu = S_r$ cannot occur.

Thus, for each $W^\nu \neq 0$, $W^\nu/W^{\nu+1}$ is both the left and right socle of $R/W^{\nu+1}$ and Lemma 3.1 implies that $\partial_l(W^\nu/W^{\nu+1}) \leqslant 2$. The rest of Lemma follows easily.

The following lemma is a slight modification of the result of Osofsky [12] that the radical W of a right perfect ring is nilpotent if the left R-module W/W^2 is artinian. For the convenience of the reader we shall repeat briefly the proof.

LEMMA 3.4. *Let R be a ring with bi-T-nilpotent radical W such that the left R-module W/W^2 is finitely generated. Then there exists n_0 such that $W^{n_0} = W^{n_0+1}$.*

Proof. Since W/W^2 is an artinian left R-module, it is finitely generated; thus, there is a finite number of elements $w_i \in W$ with

$$W/W^2 = \sum_i R\bar{w}_i, \qquad \text{where} \qquad \bar{w}_i = w_i + W^2.$$

For every natural number n, denote by A_n the set of all possible products $w_{i_1}w_{i_2} \cdots w_{i_n} \neq 0$. A path of length n is defined to be a set $\{a_k \mid a_k \in A_k$ for $1 \leqslant k \leqslant n\}$ such that

$$a_k = a_{k-1}w_{i_k} \qquad \text{for an even } k, \text{ and}$$

$$a_k = w_{i_k}a_{k-1} \qquad \text{for an odd } k \geqslant 3.$$

Now, assuming that there are arbitrary large n with $A_n \neq \varnothing$, that is to say, that there are paths of arbitrarily large length n, we can apply König's Graph

Theorem and deduce that there is a path of infinite length. But this contradicts the fact that W is bi-T-nilpotent. We conclude that there is an integer n_0 such that all products $w_{i_1} w_{i_2} \cdots w_{i_{n_0}}$ equals zero.

In order to complete the proof, we are going to show that $W^{n_0} \subseteq W^{n_0+1}$. First, observe that W^{n_0} is generated by W^{n_0+1} and the elements of the form

$$z = \rho_1 w_{i_1} \rho_2 w_{i_2} \cdots \rho_{n_0} w_{i_{n_0}} .$$

Since W/W^2 is an $R-R$-bimodule, $\bar{w}_{i_n} \rho = \sum_i \rho_i' \bar{w}_i$ for suitable $\rho_i' \in R$. Consequently, modulo W^{n_0+1}, we can pull each ρ_i occuring in z past all of the w_{i_n} and obtain that z is equal modulo W^{n_0+1} to a sum of elements of the form $\rho w_{j_1} w_{j_2} \cdots w_{j_{n_0}}$. But all these products are zero, and therefore $z \in W^{n_0+1}$, as required.

Now, we are ready to prove Theorem A (see the introduction).

Proof of Theorem A. Let R be a left balanced ring with the radical W. Hence, in view of Theorem 16 of Camillo in [1], R/W^2 is a left balanced semiprimary ring. Therefore, according to a theorem of Fuller [6], R/W^2 is a direct sum of finitely many full matrix rings over left-balanced local rings R_i. Let W_i be the radical of R_i. Since the radical of R/W^2 is nilpotent, W_i is nilpotent, as well. Thus, Lemma 3.3 implies that R_i are left artinian. Therefore, R/W^2 is left artinian and, in particular, W/W^2 is an artinian left R-module. Since W is bi-T-nilpotent (Propositions 15 and 16 of Camillo in [1]), we can apply Lemma 3.4 and obtain $W^{n_0} = W^{n_0+1}$ for some n_0.

Now, assume that R is not right perfect. Then, there is a quotient ring R' without minimal left ideals (see, e.g., Dlab [4]). But R' is left-balanced, and thus the above considerations apply to R': denoting by W' the radical of R', we have $W'^n = W'^{n+1}$ for some n. Moreover, $W'^n \neq 0$ because R' has no minimal left ideals. Consequently, the left R-module W'^n has no minimal submodule and therefore, in view of Proposition 16 of Camillo in [1], there is a left ideal L' of R' which is a maximal submodule of W'^n. But L' must contain W'^{n+1} which implies that $W'^n \neq W'^{n+1}$. This contradiction establishes that R is right perfect.

However, a right perfect ring has no nonzero idempotent ideals contained in its radical and thus, $W^{n_0} = W^{n_0+1}$ yields $W^{n_0} = 0$. This means that R is semiprimary and another application of the above mentioned theorem of Fuller and Lemma 3.3 completes the proof of Theorem A.

4. Further Constructions of Nonbalanced Modules

The following additional constructions will be needed in the final Sections 5 and 6.

CONSTRUCTION V. *Let M be an indecomposable left R-module of finite length. Assume that M possesses a proper submodule and a quotient both isomorphic to a faithful R-module N. Then M is a faithful R-module which is not balanced.*

Proof. Since M is an indecomposable module of finite length, the centralizer $\mathscr{C}$ is a local ring; denote its radical by $\mathscr{W}$. Let ι be an embedding $\iota: N \to M$. First, let us show that $N\iota \subseteq M\mathscr{W}$. If $\epsilon: M \to N$ is an epimorphism, then obviously $\epsilon\iota \in \mathscr{C}$. But $N\iota$ is a proper submodule of M, and therefore $\epsilon\iota$ is not a unit, i.e., belongs to $\mathscr{W}$. Consequently, $N\iota = M\epsilon\iota \subseteq M\mathscr{W}$.

Now, both $M/M\mathscr{W}$ and $\operatorname{soc}(M_{\mathscr{C}})$ are nontrivial* and thus, there exists a nonzero $\mathscr{C}$-homomorphism

$$\Psi': (M/M\mathscr{W})_{\mathscr{C}} \to \operatorname{soc}(M_{\mathscr{C}}).$$

As a consequence, the morphism $\Psi = \iota'\Psi'\epsilon'$ (with the canonical epimorphism $\epsilon': M_{\mathscr{C}} \to M/M\mathscr{W}$ and the embedding $\iota': \operatorname{soc}(M_{\mathscr{C}}) \to M_{\mathscr{C}}$) belongs to the double centralizer of M. But Ψ is not induced by ring multiplication. For, assuming that

$$\Psi(m) = \rho m \qquad \text{for all} \quad m \in M$$

with a suitable $\rho \in R$, we see immediately that $\rho \neq 0$ (since $\Psi \neq 0$) and that

$$\rho N\iota \subseteq \rho(M\mathscr{W}) = \Psi(M\mathscr{W}) = 0.$$

However, this contradicts the fact that N is faithful. It follows that M is not balanced.

The preceding result will be used in the next Construction VI. There, as well as in Construction VII, the double centralizer of an indecomposable module will be explored. To simplify the presentation, the following lemma will be found useful.

LEMMA 4.1. *Let R be a local ring with the radical W. Let x, y and z be elements of R such that*

$$x \neq 0, \qquad xW = 0,$$

$$y \notin zW, \qquad Wy = 0 \qquad \text{and} \qquad z \notin Rx + yR.$$

Then

$$M = (R \oplus R)/D \qquad \text{with} \quad D = \{(\kappa y, -\kappa z + \lambda x) \mid \kappa, \lambda \in R\}$$

is a faithful indecomposable left R-module. Moreover, if $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$, $\alpha_{ij} \in R$, represents an endomorphism of the left R-module $R \oplus R$ which maps D into D, then $\alpha_{21} \in W$.

* Compare N. Bourbaki, Algèbre, Ch. 8, Modules et anneaux semisimples, Ex. 3, pp. 26–27.

Proof. Put $T = (W \oplus R)/D$. The submodule T can be characterized in the following way

$$T = \{m \mid m \in M \text{ with } \operatorname{ann}(m) \neq 0\}.$$

For, take $m = (w, r) + D$ with $w \in W$, $r \in R$, and consider two cases. If $r \in W$, then $\operatorname{soc}(R_R) \subseteq \operatorname{ann}(m)$. If r is a unit, then $0 \neq xr^{-1} \in \operatorname{ann}(m)$, because

$$xr^{-1}m = (xr^{-1}w, x) + D = (0, x) + D = 0 \in M.$$

Conversely, if $\operatorname{ann}(m) \neq 0$ for some $m = (r_1, r_2) + D \in M$, then

$$(\mu r_1, \mu r_2) = (\kappa y, -\kappa z + \lambda x) \text{ for some } \mu \neq 0, \kappa \text{ and } \lambda \text{ of } R.$$

Assuming that r_1 is a unit, we get $\mu = \kappa y r_1^{-1}$ and thus

$$\kappa y r_1^{-1} r_2 = -\kappa z + \lambda x;$$

however, since $\mu \neq 0$, κ is necessarily a unit, and therefore

$$z = \kappa^{-1}\lambda x + y(-r_1^{-1} r_2),$$

a contradiction of $z \notin Rx + yR$.

Now, $m_0 = (1, 0) + D \notin T$. This follows easily from the preceding characterization of T. For, if $\mu m_0 = 0$ for some $\mu \neq 0$, then

$$(\mu, 0) = (\kappa y, -\kappa z + \lambda x) \qquad \text{for suitable} \quad \kappa, \lambda \in R$$

and thus κ is a unit. But then $z = \kappa^{-1}\lambda x$, a contradiction.

Let us assume that M is not indecomposable. First, as a consequence of this assumption, we are going to establish the fact that Rm_0 is a direct summand of M. Indeed, since $\partial(M/\operatorname{rad} M) \leqslant 2$, M is the direct sum of two local modules; write

$$M = Ra \oplus Rb \quad \text{with suitable} \quad a, b \in M.$$

Thus, $m_0 = \rho_1 a + \rho_2 b$ for some $\rho_1, \rho_2 \in R$, and since $m_0 \notin T$, we can assume that $\operatorname{ann}(\rho_1 a) = 0$. But, this implies that $Rm_0 \cap Rb = 0$; for, if $\sigma m_0 = \tau b$, then $\sigma\rho_1 a = -\sigma\rho_2 b + \tau b$, and thus $\sigma = 0$. Moreover, ρ_1 is a unit; otherwise, we would have $\operatorname{soc}(R_R) \subseteq \operatorname{ann}(\rho_1) \subseteq \operatorname{ann}(\rho_1 a)$. Hence $a = \rho_1^{-1} m_0 - \rho_1^{-1}\rho_2 b$ and we conclude that $M = Rm_0 \oplus Rb$.

Now, denote by ϵ the canonic epimorphism $\epsilon\colon M \to Rm_0$. Clearly, $m_0\epsilon = m_0$. Let $\eta\colon R \to M$ be given by $1\eta = (0, 1) + D$.

$$z\eta = (0, z) + D = (y, 0) + D = y m_0.$$

Furthermore, let $\xi: R \to Rm_0$ be given by $1\xi = m_0$. Obviously, ξ is an isomorphism. Consequently, the monomorphism

$$R \xrightarrow{\eta} M \xrightarrow{\epsilon} Rm_0 \xrightarrow{\xi^{-1}} R$$

maps z into y, and since it has to be induced by right multiplication, we get that $y = z\rho$ for a suitable $\rho \in R$. However, this is impossible: $\rho \in W$ implies $y \in zW$, while $\rho \notin W$ implies $z = y\rho^{-1} \in yR$, a contradiction in both instances. Therefore, M is indecomposable.

Finally, let $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ be an endomorphism of ${}_R(R \oplus R)$ mapping D into D. Then,

$$(0, x)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = (x\alpha_{21}, x\alpha_{22}) = (\kappa y, -\kappa z + \lambda x) \qquad \text{for suitable} \quad \kappa, \lambda \in R.$$

Assuming that $\alpha_{21} \notin W$, we get $x = \kappa y \alpha_{21}^{-1}$ and thus $\kappa \notin W$. Therefore, $z = \kappa^{-1}\lambda x + y(-\alpha_{21}^{-1}\alpha_{22}) \in Rx + yR$, contrary to our assumption. The proof of our lemma is completed.

CONSTRUCTION VI. *Let R be a left artinian local ring. Denote by S the intersection of the left and the right socles of R. Let x and y be two elements of S such that Rx and Ry are not two-sided ideals and S is not equal to $Rx + yR$. Then there exists a finitely generated faithful left R-module which is not balanced.*

Proof. If the finitely generated faithful R-modules R/Rx and R/Ry are not isomorphic, then every homomorphism between them maps one into the radical of the other and Construction II applies. Hence, we assume that $R/Rx \simeq R/Ry$.

Take an element $z \in S\backslash(Rx + yR)$ and consider the finitely generated faithful R-module $M = (R \oplus R)/D$ of Lemma 4.1. Since R is left artinian, M is an indecomposable module of finite length.

Also, observe that $N = R/Rx$ is a faithful R-module isomorphic both to $X = (Ry \oplus R)/D$ and to M/X. The first assertion follows from the fact that the map $R \to X$ defined by sending 1 into $(0, 1) + D \in X$ is surjective and has Rx as its kernel, the other is a consequence of our hypothesis $R/Rx \simeq R/Ry : M/X \simeq (R \oplus R)/(Ry \oplus R) \simeq R/Ry \simeq N$. As a result, we can apply Construction V and complete the proof.

CONSTRUCTION VII. *Let R be a left artinian local ring. Denote by S the intersection of the left and the right socles of R. Let x be a nonzero element of S such that Rx is a two-sided ideal. Furthermore, let y and z be two elements of R such that*

$$y \notin Rx, \qquad Wy = 0, \qquad yW \subseteq Rx$$

and

$$z \notin Rx + yR, \qquad Wz + zW \subseteq Rx.$$

Then there exists a finitely generated faithful left R-module which is not balanced.

Proof. Consider the finitely generated faithful R-module $M = (R \oplus R)/D$ of Lemma 4.1. Let $\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}$ represent an endomorphism of the left R-module $R \oplus R$ mapping D into D and let the induced endomorphism φ of M be nilpotent; let, $\varphi^n = 0$. Under this hypothesis, we can show that, in addition to $\alpha_{21} \in W$ (Lemma 4.1), also α_{11}, α_{12} and α_{22} belong to W.

First,

$$(x, 0)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = (x\alpha_{11}, x\alpha_{12})$$

and, since Rx is a two-sided ideal,

$$(x\alpha_{11}, x\alpha_{12}) + D = (x\alpha_{11}, 0) + D.$$

By induction, one gets immediately that

$$(x, 0)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}^n + D = (x\alpha_{11}^n, 0) + D.$$

Thus, $(x\alpha_{11}^n, 0) \in D$, i.e., $x\alpha_{11}^n = \kappa y$ for a suitable $\kappa \in R$. Therefore, $\alpha_{11} \in W$; for, otherwise, $\kappa \notin W$ and $y = \kappa^{-1}x\alpha_{11}^n \in Rx$, a contradiction.

Now, we show that $\alpha_{22} \in W$. First, we get, for arbitrary $\mu_k \in R$,

$$\begin{aligned}(\mu_k x, y\alpha_{22}^k)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} &= (\mu_k x\alpha_{11} + y\alpha_{22}^k\alpha_{21}, \mu_k x\alpha_{12} + y\alpha_{22}^{k+1})\\ &= (y\alpha_{22}^k\alpha_{21}, \mu_k x\alpha_{12} + y\alpha_{22}^{k+1})\\ &= (\mu_{k+1}x, \mu_k x\alpha_{12} + y\alpha_{22}^{k+1}) \qquad \text{for a suitable} \quad \mu_{k+1} \in R,\end{aligned}$$

because $\alpha_{11} \in W$ and $\alpha_{21} \in W$. Hence,

$$(\mu_k x, y\alpha_{22}^k)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} + D = (\mu_{k+1}x, y\alpha_{22}^{k+1}) + D.$$

Therefore, by induction,

$$(0, y)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}^n + D = (\mu_n x, y\alpha_{22}^n) + D \qquad \text{for some} \quad \mu_n \in R.$$

We deduce that

$$(\mu_n x, y\alpha_{22}^n) = (\kappa y, -\kappa z + \lambda x) \qquad \text{for suitable} \quad \kappa, \lambda \in R.$$

Necessarily, $\kappa \in W$ and thus $y\alpha_{22}^n \in Wz + Rx \subseteq Rx$; therefore, $\alpha_{22} \in W$.

Finally, calculate

$$(y, -z)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = (y\alpha_{11} - z\alpha_{21}, y\alpha_{12} - z\alpha_{22})$$
$$= (\kappa y, -\kappa z + \lambda x) \qquad \text{for suitable} \quad \kappa, \lambda \in R.$$

Since $\alpha_{11} \in W$ and $\alpha_{21} \in W$, $y\alpha_{11} - z\alpha_{21} \in yW + zW \subseteq Rx$. Again, this implies that $\kappa \in W$, and thus

$$y\alpha_{12} \in zW + Wz + Rx \subseteq Rx;$$

therefore $\alpha_{12} \in W$, as required.

Now, consider the module M over its centralizer $\mathscr{C}$; let us remark that $\mathscr{C}$ is a local ring with a nil radical $\mathscr{W}$. Again, we can easily show that $(x, 0) + D$ belongs to $\operatorname{soc}(M_{\mathscr{C}})$. For, if $\varphi \in \mathscr{W}$, then $\varphi^n = 0$ and taking

$$\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} \in \operatorname{end}_R(R \oplus R)$$

which induces φ, we have

$$(x, 0)\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix} = (x\alpha_{11}, x\alpha_{12}) = (0, 0);$$

here, $(x, 0) + D$ is obviously a nonzero element of M. Furthermore, since all $\alpha_{ij} \in W$,

$$X = (W \oplus W)/D \subseteq M\mathscr{W}.$$

This enables us to define the following element Ψ of the double centralizer of M:

$$\Psi = M_{\mathscr{C}} \xrightarrow{\epsilon} M_{\mathscr{C}}/X \xrightarrow{\Psi'} \operatorname{soc}(M_{\mathscr{C}}) \xrightarrow{\iota} M_{\mathscr{C}}$$

with the canonic epimorphism ϵ, the embedding ι and Ψ' such that $\Psi'[(0, 1) + X] = (x, 0) + D$. This morphism cannot be induced by ring multiplication. For, if

$$\Psi[(0, 1) + D] = \rho[(0, 1) + D] \text{ for a suitable } \rho \in R,$$

then

$$(x, 0) + D = (0, \rho) + D;$$

hence, $(x, -\rho) \in D$. Since $Rx \cap Ry = 0$, this implies readily that x must be equal to 0, a contradiction. The proof of Construction VII is completed.

5. Structure of Left-Balanced Local Rings

We start with the following lemma, which extends the investigations of the left $QF-1$ rings in Section 3. This lemma will be used to establish a structural characterization of the left balanced local rings in the present section, as well as to exclude the case (iii) of Lemma 3.1 for rings which are finitely generated over their centers in the next section.

Lemma 5.1. *Let R be a left artinian left $QF-1$ local ring. Then, for any two nonzero elements x and y which belong to the intersection S of the left and the right socles, we have the equality*

$$Rx + yR = S.$$

Proof. In view of Lemma 3.1, S is a minimal two-sided ideal and $\partial_l(S) \leqslant 2$. Let x and y be nonzero elements of S. If $\partial_l(S) = 1$, then $Rx = S$. If $\partial_l(S) = 2$, then Rx is not a two-sided ideal, and thus the equality $Rx + yR = S$ follows immediately from Construction VI.

Now, let R be a local ring and consider the quotient rings R/W^2 and R/W^3. First, W/W^2 is both the left and the right socle of R/W^2. Therefore, if R/W^2 is left balanced, it turns out by Lemma 3.1 that W/W^2 is the unique minimal two-sided ideal of R/W^2 and that $\partial_l(W/W^2) \leqslant 2$. If $\partial_l(W/W^2)$ equals 0 or 1, then R/W^2 is left uniserial and, as a consequence, R itself is left uniserial. Hence, we may restrict our attention to the case when $\partial_l(W/W^2) = 2$. It is shown in our paper [5] that this case can actually happen.

Now, let us strengthen our assumption and consider the case when R/W^3 is left balanced. Again, according to Lemma 3.3, W^2/W^3 is both the left and the right socle of R/W^3 and $\partial_l(W^2/W^3) \leqslant 2$. We are going to show that, in fact, $\partial_l(W^2/W^3)$ equals either 0 or 1.

Lemma 5.2. *Let R be a local ring such that R/W^3 is left balanced. Then $\partial_l(W^2/W^3) \leqslant 1$.*

Proof. Obviously, we may assume that $W^3 = 0$. Applying Lemma 5.1 to the ring R/W^2, we can see that, for any element $w \in W \backslash W^2$, we have the equality

$$Rw + wR + W^2 = W.$$

It has been proved in Lemma 3.3 that W^2 is both the left and the right socle of R. Thus, in particular, both Rw and wR intersect W^2 nontrivially. Taking nonzero elements $x \in Rw \cap W^2$ and $y \in Rw \cap W^2$ and making use of Lemma 5.1, we deduce that

$$Rx + yR = W^2.$$

Therefore, $Rw + wR \supseteq W^2$ and we conclude that

$$Rw + wR = W \qquad \text{for all} \quad w \in W \backslash W^2.$$

Now, if we assume that $\partial_l(W^2) = 2$, then also $\partial_l(W/W^2) = 2$, and therefore $\partial_l(W) = 4$ and $\partial_l(R) = 5$. Let w be an element of R with $\partial(Rw) = 2$. Hence, $\operatorname{ann}_l(w)$ has length 3, and therefore is not contained in W^2. Let us take an element $v \in W \backslash W^2$ with $vw = 0$. Thus we have the equalities

$$Rv + vR = W \qquad \text{and} \qquad Rw + wR = W,$$

and consequently

$$Ww = (Rv + vR)w = vRw = v(Rw + wR) = vW$$

implying that Ww is a two-sided ideal. Therefore, since $Ww \subseteq W^2$ and since W^2 is a minimal two-sided ideal, either $Ww = 0$ or $Ww = W^2$. But $Ww \neq 0$ because $\partial[\operatorname{ann}_l(w)] = 3$ and $Ww \neq W^2$ because $Rw \not\supseteq W^2$. This shows that $\partial_l(Rw) \neq 2$ for any $w \in W$.

Next, we show that the faithful left R/W^2-module $M_1 = W$ has no monogenic quotient of length 2. For, otherwise the kernel would be of length 2, and since there is no monogenic submodule of length 2, it would equal to W^2; however, this is impossible because W/W^2 is not monogenic.

Thus, denoting by M_2 a monogenic left R-module of length 2, we can check easily that M_1 and M_2 are faithful R/W^2-modules which satisfy the assumptions of Construction II. This contradiction establishes $\partial_l(W^2) \leqslant 1$, as desired.

Now, we are ready to give

Proof of Theorem B. Assume that R is not left uniserial. Then, in view of Lemma 3.3, $\partial_l(W/W^2) = 2$. If $W^2 = 0$, R is of the type described in (ii); this follows immediately from Lemma 5.1. If $W^2 \neq 0$, we get, according to Lemma 5.2, that $\partial_l(W^2/W^3) = 1$. Furthermore, by Lemma 3.3, W^2/W^3 is the left socle of R/W^3. From here, we deduce easily that $W^3 = 0$. Indeed, taking $w \in W \backslash W^2$ and assuming $W^3 \neq 0$, necessarily $\partial(Rw) \geqslant 3$ and Rw has a uniserial quotient of length 3. Now, $Rw \simeq R/\operatorname{ann}_l(w)$; however, R has obviously no uniserial quotient of length 3. The proof of Theorem B is completed.

Remark. It is easy to see that in the case (iii) of Theorem B, R cannot be right uniserial. For, in such a case, we would have

$$\partial_l(\operatorname{soc}{}_R R) = \partial_r(\operatorname{soc} R_R) = 1,$$

and R would be a quasi-Frobenius ring. But this is impossible, because $\partial_l(R) = 4$ while $\partial_r(R)$ would equal 3.

An immediate consequence of Theorem A and Theorem B is the following

COROLLARY 5.3. *Let R be a left-balanced local ring. Then R is of the type* (i) *or* (ii) *or* (iii) *of Theorem B. Thus a left-balanced ring is a finite direct sum of full matrix rings over rings of the types* (i) *or* (ii) *or* (iii) *of Theorem B.*

If R is both left and right balanced, we get the following result.

COROLLARY 5.4. *Let R be a balanced local ring with the radical W. Then either*

(i) *R is uniserial; or*

(ii) $W^2 = 0$, $\partial_l(W) \leqslant 2$, $\partial_r(W) \leqslant 2$ *and, for any two nonzero elements x, y of W, $Rx + yR = W$; or*

(iii) *R is a quasi-Frobenius ring of length* 4, *and for any two elements x, y of $W \backslash W^2$, $Rx + yR = W$.*

6. RINGS FINITELY GENERATED OVER THEIR CENTERS

In this final section, we are going to show that Jans' conjecture is true for rings which are finitely generated over their centers. We start with a result on $QF - 1$ rings. Let us recall that, by Lemma 3.1, a perfect left $QF - 1$ local ring has a unique two-sided ideal which is either the left socle or the right socle of it.

LEMMA 6.1. *Let R be a perfect left $QF - 1$ local ring. Assume that R is finitely generated over its center. Then the unique minimal two-sided ideal is both a minimal left ideal and a minimal right ideal.*

Proof. Let I be the unique minimal two-sided ideal. Thus, I can be considered as an $R - R$-bimodule. Furthermore, since $I \subseteq \mathrm{soc}({}_RR) \cap \mathrm{soc}(R_R)$, we have $WI = 0 = IW$; it turns out that I is, in fact, an $R/W - R/W$-bimodule.

Now, let Z be the center of R. Hence, $(Z + W)/W$ is contained in the center of the division ring R/W, and we can consider the quotient field F of $(Z + W)/W$ as a subring of R/W. It is immediate from the $R/W - R/W$-bimodule structure of I that the equality

$$\kappa x = x\kappa$$

holds for all $\kappa \in F$ and all $x \in I$.

We have assumed that R is finitely generated as a Z-module. Hence, R/W is finitely generated as a $(Z + W)/W$-module and is therefore a finite-dimensional vector space over F. Let n be the dimension $\dim({}_F R/W)$. If $\dim({}_{R/W}I) = m$, then $\dim({}_F I) = mn$ and, obviously, this does not depend on whether we consider the left or the right action of F on I. Consequently,

$$\dim({}_{R/W}I) = \dim(I_{R/W}).$$

Now, in view of Lemma 3.1, we know that $\partial_1(I) = \dim({}_{R/W}I)$ equals 1 or 2 and thus to prove our lemma, it is sufficient to show that the case $\dim({}_{R/W}I) = 2$ cannot occur. Supposing the contrary and applying Lemma 5.1, we know that

$$Rw + wR = I \qquad \text{for all} \quad w \neq 0 \text{ in } I.$$

Therefore,

$$\begin{aligned} 2n &= \dim_F I = \dim_F(Rw + wR) \\ &= \dim_F Rw + \dim_F wR - \dim_F(Rw \cap wR) \\ &= n + n - \dim_F(Rw \cap wR). \end{aligned}$$

Since, obviously, $\dim_F(Rw \cap wR) \geqslant 1$, we get a contradiction and therefore $\dim({}_{R/W}I) = 1$, as required.

As a result of the previous lemma, we are ready to present

Proof of Theorem C. If R is uniserial, then it is left balanced. On the other hand, assume that R/W^2 is left balanced. Then, R/W^2 is a finite direct sum of full matrix rings over left-balanced local rings R_i (cf. Fuller [6]). Of course, R_i are finitely generated over their centers. Let W_i be the radical of R_i. Evidently, $W_i^2 = 0$. Thus, W_i is the left and the right socle of R_i. By Lemma 3.1 and Lemma 6.1, W_i is both a minimal left and a minimal right ideal of R_i, and therefore R_i is uniserial. Consequently, R/W^2 is uniserial. And, by a simple argument due to Nakayama [9], R is necessarily uniserial, as well.

Remark 1. Let us point out that on the basis of Theorem C the following conditions can easily be shown to be equivalent for a ring R finitely generated over its center:

(i) *R is uniserial,*

(ii) *R is left balanced,*

(ii)* *R is right balanced,*

(iii) *R/W^2 is left balanced,*

(iii)* *R/W^2 is right balanced.*

Note that this remark applies, in particular, to finite-dimensional algebras over arbitrary fields and to finite rings.

Remark 2. The fact that R/W^2 is left balanced does not imply in general that the ring R itself is left balanced. An example to that effect is given in [5].

In order to prove Theorem D, we need a result asserting that, for left artinian rings, the case (ii) of Lemma 3.1 can take place only if the left and the right socles coincide.

LEMMA 6.2. *Let R be a left artinian left $QF - 1$ local ring. Then the left socle of R is the unique minimal two-sided ideal.*

Proof. According to Lemma 3.1, we can assume that the right socle S_r of R is properly contained in the left socle S_l of R and that $\partial_l(S_r) = 1$. We want to show that the intersection of the left and the right socles of R/S_r is contained in S_l/S_r. Indeed, choose first a nonzero element $x \in S_r$ (thus, $Rx = S_r$ is a two-sided ideal) and an element $y \in S_l \backslash S_r$ such that $y + S_r$ belongs to the right socle of S_l/S_r (thus, $y \notin Rx$, $Wy = 0$ and $yW \subseteq Rx$). Then, if z is an arbitrary element such that $z + S_r$ belongs both to the left and the right socles of R/S_r (that is, if $Wz + zW \subseteq Rx$), necessarily $z \in Rx + yR \subseteq S_l$ in view of Construction VII.

Now, if $W^n \neq 0$ and $W^{n+1} = 0$, obviously $S_r = W^n$; this follows from the fact that S_r is the unique minimal two-sided ideal. Moreover, W^{n-1} must be contained in S_l, because W^{n-1}/W^n is contained in the intersection of the left and the right socles of R/W^n which, in turn, is contained in S_l/S_r, as shown above. Hence,

$$W^n = W \cdot W^{n-1} \subseteq W \cdot S_l = 0,$$

contradicting our hypothesis.

Now, it is easy to give

Proof of Theorem D. Applying Lemma 6.2 both to the left and the right of the ring R, we get immediately that $\mathrm{soc}({}_RR) = I = \mathrm{soc}(R_R)$ is the unique minimal two-sided ideal. And, by Lemma 6.1, it follows that $\partial_l(I) = \partial_r(I) = 1$. This completes the proof of Theorem D.

Remark 3. It should be mentioned that the conclusion of Theorem D can be immediately extended to finite direct sums of full matrix rings over artinian local rings. For, such a ring is $QF - 1$ or quasi-Frobenius if and only if the local rings involved are $QF - 1$ or quasi-Frobenius, respectively.

Remark 4. The assumption on R to be finitely generated over its center is necessary both in Theorem C and Theorem D. For, it has been shown in [5] that there exists a class of artinian local rings which are balanced but not quasi-Frobenius.

After completion of this paper, the authors have learnt that Camillo and Fuller have proved independently Theorems C and D for finite-dimensional algebras.

Added in Proof (June, 1972). A full characterization of balanced rings is given in "Lecture Notes in Mathematics," p. 246, Springer-Verlag, New York, 1972.

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